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ON FRACTURE OF VISCOELASTIC BODIES*

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On the basis of a thermodynamic variational principle, a criterion for quasistatic crack growth in a viscoelastic solid is deduced: the change in the sum of the scattering and the rate of decrease of elastic energy equals the increment in surface dissipation. The appropriate invariant contour integral is found. The theory proposed is suitable for cracks of any shape for any loading path on viscoelastic solids. Examples are considered, including crack growth with localized viscous dissipation.

1. For the system of a solid with a crack subjected to external effects the following energy balance can be formed

$$P\Delta^{\cdot} + Q = \mathrm{E}^{\cdot} \tag{1.1}$$

where P is the external effect, $\Delta^{\cdot}$ is the corresponding displacement, Q is the heat influx, and E is the internal energy and differentation is with respect to time. The equation of entropy S growth has the form

$$TS^{\cdot} = T\Pi + Q \tag{1.2}$$

where T is the temperature, and Π is the rate of entropy production.

Elimination of the heat influx Q permits expressing Π in terms of the free energy of the solid $\Phi = \mathrm{E} - TS$ in the following way in the isothermal case $(T = T_0)$:

$$\Pi = (P/T_0)\Delta^{\cdot} + S^{\cdot} - (\mathrm{E}^{\cdot}/T_0) = (P/T_0)\Delta^{\cdot} - (\Phi^{\cdot}/T_0) \tag{1.3}$$

On the other hand, the dissipation function Ψ can be introduced, and the variational principle of least energy dissipation /1/ requires that

$$\delta(\Pi - \Psi) = 0 \tag{1.4}$$

where we perform the variation with respect to the forces P and the crack length l.

Further manipulations are associated with the spearation of specific energies expended in the formation of fracture surfaces. Thus the surface energy $2\gamma_0 l$ is separated /2/ in the free energy Φ:

$$\Phi = \Phi(l, P) = W(l, P) + 2\gamma_0 l, \quad \gamma_0 = \gamma_0(T_0, a) \tag{1.5}$$

where a is the linear scale (thickness) of the surface zone of the material, and W is the elastic energy of a solid with a slit of length l evaluated by means of the specific volume /3/ energy U, where G is the Irwin force

$$\frac{\partial\Phi}{\partial P} = \frac{\partial W}{\partial P} = \Delta^e, \quad \frac{\partial W}{\partial l} = -G \tag{1.6}$$

In contrast to the analysis /2/, the electric part Δ^e of the total displacement $\Delta = \Delta^e + \Delta^p$ is also separated here and W is given by functions of the forces P.

We shall give the dissipation Ψ as follows

$$T_0\Psi = \Lambda(P, l) + 2\gamma_* l^{\cdot} + 2\xi l \tag{1.7}$$

Here Λ is the dissipation in a solid with a slit calculated by means of the specific volume /3/ dissipation D, $2\gamma_* l^{\cdot}$ is the rate of mechanical energy dissipation due to the rate of crack growth $l^{\cdot}$, but independent of its length, $2\xi l$ is the specific energy dissipation in the surface layer, where $\xi = \xi(l^{\cdot})$ can depend on the rate of crack growth.

The condition of autonomy of crack growth is that γ_0, γ_*, ξ are independent of P.

Substitution of (1.5) and (1.7) into (1.4) results in the following:

$$\delta(\Lambda + \partial W/dt) - \Delta^{\cdot}\delta P = -2\xi\delta l - 2\delta(\gamma l^{\cdot}) \tag{1.8}$$

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where we consider the variation $\delta(\gamma l^{\cdot}) = 0,\ \gamma = \gamma_0 + \gamma_*$.

The quantity 2ξ is interpreted /3/ as the specific mechanical energy dissipation per unit length of the surface layer due to the special surface viscosity of the material (near the past crack). Its magnitude is estimated, respectively,

$$\xi = \frac{[\mu]}{a}\left(\frac{a}{\tau}\right)^2 f\left(\frac{l^{\cdot}\tau}{a}\right) = \frac{[\mu]}{\mu}\frac{\gamma E}{\mu} f\left(\frac{l^{\cdot}\mu}{\gamma}\right), \quad \tau = \frac{\mu}{E}, \quad a = \frac{\gamma}{E} \tag{1.9}$$

Here $[\mu] = \mu_s - \mu$ is the jump in the dynamic viscosity on the interfacial surface, μ_s is the surface viscosity, μ is the viscosity of the same material in bulk, E is Young's modulus, and τ is the relaxation time.

The concept of special surface viscosity is known in hydrodynamics. It was introduced to take account of the effect of an adsorption film /4/. The quantity 2ξ was clarified in /3/ as resistance to crack growth in an idealized model of viscous solid fracture. It is expedient to use the surface viscosity and surface dissipation rate 2ξ even in the mechanics of viscoelastic media. For instance, for such media as polymers, the fact of rearrangement of the internal structure near a fracture surface has been well established /5/. Energy, usually taken into account as the surface energy γ_0, where the introduction of γ_* means taking account of viscous dissipation that occurs during this rearrangement, is expended in this rearrangement. It is also natural to assume that the strip of material with rearranged structure dissipates mechanical energy differently during viscoelastic strain than prior to the rearrangement.

We present a numerical estimate of the surface dissipation by the formula $\xi \sim \gamma E/\mu$. We have $\xi \sim 10^9 - 10^6$ N.m^{-1}.s^{-1} for the values $\gamma \sim 10^2$ N/m, $E \sim 10^8$ Pa, $\mu \sim 10 - 10^4$ Pa.s which are characteristic for polymers. Hence $\xi l \gg \gamma l^{\cdot}$ under the completely real condition $(10^4 - 10^7)\, l \gg (l^{\cdot} \cdot c)$ for the quasistatic growth of small cracks. The surface layer thickness here is $a \sim 10^{-6}$ m, $\tau \sim 10^{-7} - 10^{-4}$ s, and the characteristic flow velocity in the layer is $a/\tau \sim (10 - 10^{-2})$ m/s.

The variational equation (1.8) decomposes into two independent relations, of which the first governs the inelastic part of the displacement velocity

$$\frac{\partial\Lambda}{\partial P} = \Delta^{\cdot} - \frac{\partial W}{\partial P} = \Delta^{\cdot} - \Delta^{\cdot e} = \Delta^{\cdot p}$$

while the second yields a criterion for crack growth in a viscoelastic solid

$$N \equiv -\frac{\partial}{\partial l}\left(\Lambda + \frac{\partial W}{\partial t}\right) \equiv -\frac{\partial\Lambda}{\partial l} + \frac{\partial G}{\partial l} l^{\cdot} = 2\xi \tag{1.10}$$

In other words, a crack grows if the sum of changes in the dissipation and the rate of decrease in the potential energy of the solid reaches the critical level specified by the right side in (1.10).

2. For viscoelastic (creep) media it is customary to introduce the function $T_0\Psi^\circ$ such that

$$e_{ij}^{\cdot} = T_0 \frac{\partial\Psi^\circ}{\partial\sigma_{ij}}, \quad T_0\Psi^\circ = 2D + \frac{\partial u}{\partial t} \tag{2.1}$$

The function $T_0\Psi^\circ$, called the additional strain power, thereby performs the role of a potential for the strain rate $e_{ij}^{\cdot}$, while the elastic energy U plays for the strain of an elastic solid. The integral over the volume V of the solid

$$T_0 \int_V \Psi^\circ dV = \Lambda + \frac{\partial W}{\partial t}, \quad W = \int U\, dV$$

is the additional strain power of the solid $T_0\Psi$ which includes the crack length l as a parameter. A stationarity variational principle, completely analogous to the principle (1.4), is formulated for the function $T_0\Psi$, and the derivation to the criterion (1.10) actually uses the method of finding the "excess" unknown l. The initial condition $l = l_0$ at $t = 0$ will be determined for the differential equation (1.10) by the solution of the elastic problem.

To analyze crack growth in a viscoelastic plane (Maxwell material) stretched by forces P, we use the customary representations

$$D = P^2/(2\mu), \quad U = P^2/(2E)$$

Since the excess values of the elastic energy and the dissipation rate are proportional to the area of stress concentration, i.e., l^2, then

$$\Lambda = \theta_1 l^2 P^2/(2\mu) + \text{const}, \quad W = -\theta_2 l^2 P^2/(2E) + \text{const} \tag{2.2}$$

where θ_1, θ_2 are certain numerical coefficients, and the constants are independent of the crack length.

Substituting the estimate (2.2) into the crack growth condition (1.10) for $\gamma = \text{const}$ results in the following differential equation

$$2\xi - \theta_1 l\,(P^2/\mu) + \theta_2 l^{\cdot}\,(P^2/E) = 0 \tag{2.3}$$

If the crack growth rate $l^{\cdot}$ does not influence the surface dissipation: $f = f_0 = \text{const}$ in the representation (1.9), then $\xi = \text{const}$ and the solution of (2.3) has the exponential form

$$l - l_* = (l_0 - l_*)\exp\left(\frac{E}{\mu}\frac{\theta_1}{\theta_2}t\right), \quad l_* = \frac{\xi\mu}{\theta_1 P^2}\left(= \frac{f_0}{\theta_1}\frac{\gamma E}{P^2}\frac{[\mu]}{\mu}\right) \tag{2.4}$$

A crack grows in a viscoelastic solid if its initial length l_0 is greater than the threshold value l_* at which the change in dissipation in the solid per unit crack length reaches the surface dissipation level. For $l_0 > l_*$ crack growth turns out to be possible because of the release of elastic energy. On the other hand, if the length l_0 is greater than l_* but less than the critical Griffith value of the length l_G, determined from the equation

$$G(l_G) \equiv \theta_2\,(P^2 l_G/E) = 2\gamma \tag{2.5}$$

then the solution (2.4) describes crack growth to the value l_G. If $l_0 = l_G$, then the finite starting rate of crack growth can be determined from equations (2.3)

$$l^{\cdot}(0) = \frac{2E^2}{\theta_2 P^2}\left(\frac{\xi\mu}{\gamma E} - \frac{\theta_1}{\theta_2}\right)\frac{\gamma}{\mu} = \frac{2E^2}{\theta_2 P^2}\left(\frac{[\mu]}{\mu}f_0 - \frac{\theta_1}{\theta_2}\right)\frac{a}{\tau} \tag{2.6}$$

Second particular case: $\xi = \zeta l^{\cdot 2}$, $\xi = [\mu]/a = \text{const}$. Then (2.3) becomes

$$\frac{d\,(l/a)}{-1 + \{1 + \beta\,(l/a)\}^{1/2}} = \alpha\,dt, \quad \alpha = \frac{\theta_2}{4}\frac{P^2}{E\,[\mu]}, \quad \beta = 8\,\frac{\theta_1}{\theta_2}\frac{E^2}{P^2}\frac{[\mu]}{\mu} \tag{2.7}$$

and its solution is

$$(-1 + \sqrt{1 + \beta\,(l/a)}) + \ln(-1 + \sqrt{1 + \beta\,(l/a)}) = {}^1/_2\alpha\beta t + \text{const} \tag{2.8}$$

The condition for applicability of the computation of a discontinuity by the viscous model /3/ follows from equation (2.3), namely, a crack should be much longer than its increment during relaxation $l \gg l^{\cdot}\tau$. For constancy of the surface dissipation such a crack grows in equilibrium only as the tensile force drops according to (2.4). For the surface dissipation dependent on the rate $l^{\cdot}$ a crack in a viscous plane grows in equilibrium even for a fixed rupturing force. The parabolic law $l \sim t^2$ /3/, which corresponds to the critical condition of viscous solid fracture

$$-\partial\Lambda/\partial l = 2\xi \tag{2.9}$$

follows from the solution (2.8) for such a situation.

In the opposite case of negligible dissipations $\Lambda = \xi = 0$, the interchangeability of the variation and time-differentiation operations reduces the variational relation (1.8) to the known representation

$$\delta W - \Delta\delta P = -2\gamma\delta l, \quad \Delta \equiv \Delta^e$$

which yields the classical result of the generalized Griffith condition (2.5)

$$G = -\partial W/\partial l = 2\gamma, \quad \gamma = \gamma_* + \gamma_0$$

The complete solution of the problem requires finding the dependence $\xi(l^{\cdot})$ for which an analysis of the fine structure of the fracture zone is needed by utilization of data on the velocity dependence of the fracture energy.

3. To find the invariant contour integral, the heat influx equation differentiated with respect to time should be considered

$$\frac{\partial L}{\partial t} = \frac{\partial}{\partial t}\left\{\frac{\partial \varepsilon}{\partial t} - \sigma_{ij}\frac{\partial e_{ij}}{\partial t} + \frac{\partial q_j}{\partial x_j}\right\} = 0, \quad L = 0 \tag{3.1}$$

where ε is the specific internal energy, σ_{ij} is the stress, q_j is the heat influx, e_{ij} is the strain, u_i is the displacement, and v_i the velocity of displacement

$$\frac{\partial e_{ij}}{\partial t}=e_{ij}^{\cdot}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right),\quad v_i=\frac{\partial u_i}{\partial t}$$

Let us go over to a moving coordinate system t' and x_j' by the rule

$$t=t',\quad x_k=x_k'+l_k^{\cdot}t,\quad \partial/\partial t'=\partial/\partial t+l_k^{\cdot}\delta_{kj}\partial/\partial x_j,\quad \partial/\partial x_k'=\partial/\partial x_k$$

Then under the assumption of stationarity of the fields in the moving coordinate system, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial \varepsilon}{\partial t}\right)=l_k^{\cdot}\frac{\partial}{\partial x_k}\left(l_n^{\cdot}\delta_{nj}\frac{\partial \varepsilon}{\partial x_j}\right)=l_n^{\cdot}\delta_{nj}\frac{\partial}{\partial x_j}\left(l_k^{\cdot}\frac{\partial \varepsilon}{\partial x_k}\right)$$

$$\frac{\partial}{\partial t}\left(\sigma_{ij}\frac{\partial e_{ij}}{\partial t}\right)=\frac{1}{2}\frac{\partial}{\partial t}\sigma_{ij}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)=\frac{\partial}{\partial t}\frac{\partial}{\partial x_j}(\sigma_{ij}v_i)=\frac{\partial^2}{\partial x_j\partial x_n}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right)l_n^{\cdot}l_k^{\cdot},$$

$$\frac{\partial}{\partial t}\left(\frac{\partial q_j}{\partial x_j}\right)=-\frac{\partial}{\partial x_j}\left(\frac{\partial q_j}{\partial x_k}\right)l_k^{\cdot}$$

where the primes have been omitted, and δ_{nj} is the unit tensor. Then equation (3.1) takes the form

$$l_n^{\cdot}\delta_{nj}\frac{\partial^2\varepsilon}{\partial x_j\partial x_k}-l_n^{\cdot}\frac{\partial^2}{\partial x_j\partial x_k}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_n}\right)=\frac{\partial^2 q_j}{\partial x_j\partial x_k}\tag{3.2}$$

Let us integrate (3.2) over the domain $A_{\beta-\alpha}$ of the stationary state between the contours Γ_α and Γ_β successively enclosing the crack vertex. This results in a contour integral of the second kind (in the terminology of /6/):

$$\int_{\Gamma_\beta}\frac{\partial}{\partial x_k}\left(l_n^{\cdot}\delta_{nj}\varepsilon-\sigma_{ij}\frac{\partial u_i}{\partial x_n}l_n^{\cdot}-q_j\right)n_j\,d\Gamma=\int_{\Gamma_\alpha}\frac{\partial}{\partial x_k}(\ldots)\cdot n_j\,d\Gamma=\mathrm{const}\tag{3.3}$$

where n_j is the normal component to the contour Γ_β. Since the heat influx equation $L=0$ is valid in the domain A_α (within the contour Γ_α directly enclosing the crack vertex), then under the usual constraints /3/ on the edges of the slit (no flux) the integral (3.3) over the contour Γ_α is zero, and therefore, the constant in the right side of (3.3) is also zero.

If the specific internal energy ε_v of the particles belonging to the bulk phase /3/ is introduced, then the integral (3.3) takes the following form:

$$I_k=\int_{\Gamma_\beta}\frac{\partial}{\partial x_k}\left(l^{\cdot}\delta_{nj}\varepsilon_v-\sigma_{ij}\frac{\partial u_i}{\partial x_n}l_n^{\cdot}-q_j\right)n_j\,d\Gamma=\int_{\Gamma_\alpha}l_n^{\cdot}\delta_{nj}\frac{\partial}{\partial x_k}(\varepsilon_v-\varepsilon)\,n_j\,d\Gamma\tag{3.4}$$

Furthermore, we consider the equation of entropy production differentiated with respect to time:

$$\frac{\partial}{\partial t}\left(T\frac{\partial s}{\partial t}\right)=-\frac{\partial}{\partial t}\frac{\partial q_j}{\partial x_j}+\frac{\partial}{\partial t}(\sigma_{ij}e_{ij}^{\cdot p})\tag{3.5}$$

In the presence of the zone $A_{\beta-\alpha}$ of stationary and isothermal states, we have in the moving coordinate system

$$\frac{\partial}{\partial t}\left(T\frac{\partial s}{\partial t}\right)=T_0 l_n^{\cdot}l_k^{\cdot}\frac{\partial^2 s}{\partial x_j\partial x_i}\delta_{nj}\delta_{ki};\quad T=T_0$$

$$\frac{\partial}{\partial t}(\sigma_{ij}e_{ij}^{\cdot p})=2\frac{\partial D}{\partial t}=-2l_k^{\cdot}\delta_{kj}\frac{\partial D}{\partial x_j}$$

Integrating (3.5) in the domain $A_{\beta-\alpha}$ results in the following invariant integral:

$$\int_{\Gamma_\beta}\frac{\partial}{\partial x_k}(T_0 l_n^{\cdot}\delta_{nj}s-q_j)\,n_j\,d\Gamma+2D\delta_{kj}n_j\,d\Gamma=2\frac{\partial\gamma_*}{\partial l_k}l_k^{\cdot}\tag{3.6}$$

where the derivative of the singular dissipation (within Γ_α), included in the integral analog of the differential relationship (3.5), is in the right-hand side. Insertion of the specific volume functions s_v and D_v reduces the contour integral (3.6) to the following:

$$F_k=\int_{\Gamma_\beta}\frac{\partial}{\partial x_k}(T_0 l_n^{\cdot}\delta_{nj}s_v-q_j)\,n_j\,d\Gamma+2D_v\delta_{kj}\,n_j\,d\Gamma=\tag{3.7}$$

$$\int_{\Gamma_\alpha}T_0 l_n^{\cdot}\frac{\partial}{\partial x_k}(s_v-s)\,\delta_{nj}n_j\,d\Gamma+2\int_{\Gamma_\alpha}(D_v-D)\,\delta_{kj}n_j\,d\Gamma+2\frac{\partial\gamma_*}{\partial l_k}l_k^{\cdot}$$

Finally, the difference between the integrals (3.4) and (3.7) results in a resultant contour integral around the crack vertex in a viscoelastic solid

$$N_k = F_k - I_k = \int_{\Gamma_\beta} \left(2D\delta_{kj} - \frac{\partial U}{\partial x_k}\delta_{kj} l_n^{\cdot} - \sigma_{ij}\frac{\partial v_i}{\partial x_k}\right) \times n_j\, d\Gamma = F_k^\circ - I_k^\circ = N_k^\circ \tag{3.8}$$

or differently

$$N_k = \int_{\Gamma_\beta} \left(2D + \frac{\partial U}{\partial t}\right) n_k\, d\Gamma - \sigma_{ij}\frac{\partial v_i}{\partial x_k} n_j\, d\Gamma = N_k^\circ \tag{3.9}$$

where k is the subscript for the axis along which the crack grows while the subscript for the volume phase is omitted. The constant N_k° is estimated (see /3/) by considering the flux through the contour Γ_α:

$$N_k^\circ = \lim_{a\to 0}\int_{-a}^{a}\left\{2[D] + \frac{\partial[f(T_0)]}{\partial t}\right\} dx_2 + 2\frac{\partial\gamma_*}{\partial t} = 2\left(\xi + \frac{\partial\gamma}{\partial t}\right) \tag{3.10}$$

$$U = \varepsilon - T_0 s, \quad \lim\,[f(T_0)]\, a = \gamma_0$$

where $[D] = D_s - D_v$ is the jump in dissipation, and $[f(T_0)] = f_s - f$ is the jump in the isothermal free potential (the elastic energy U) on the fracture surface.

As regards the quantity 2ξ, it was figured in above, and the representation (3.10) corresponds to its estimate (1.9). The result of (3.9) and (3.10) generalizes the results in /3/ directly to the case of the fracture of viscoelastic solids.

If it is furthermore assumed that the displacements velocities field is decomposable into elastic and viscous components: $v_i = v_i^e + v_i^p$, then the contour integral (3.9) is represented in the form of two components

$$N_k = -\frac{\partial\Lambda}{\partial l_k} + l_k^{\cdot}\frac{\partial G}{\partial l_k} = N_k^\circ \tag{3.11}$$

evaluated separately

$$\frac{\partial\Lambda}{\partial l_k} = -\int_{\Gamma_\beta} 2Dn_k\, d\Gamma - \sigma_{ij}\frac{\partial v_i^p}{\partial x_k} n_j\, d\Gamma$$

$$\frac{\partial G}{\partial l_k} l_k^{\cdot} = \int_{\Gamma_\beta} \frac{\partial U}{\partial t} n_k\, d\Gamma - \sigma_{ij}\frac{\partial v_i^e}{\partial x_k} n_j\, d\Gamma$$

The subscript k can be omitted for a crack being propagated rectilinearly. Then the contour integral (3.11) results in the following fracture criterion:

$$N = -\left(\frac{\partial\Lambda}{\partial l} - l^{\cdot}\frac{\partial G}{\partial l}\right) = 2\xi + 2\frac{\partial\gamma}{\partial a} a^{\cdot} \tag{3.12}$$

which agrees with the result (1.10) for an unchanged linear scale $(a = \text{const})$.

Determination of the quantities Λ, G from the solution of the problem in stresses:

$$\frac{\partial\Lambda}{\partial l} = -(\varkappa + 1)^{-1}\frac{K^2}{8\mu}, \quad G = -\frac{\partial W}{\partial l} = (\varkappa + 1)^{-1}\frac{K^2}{4E}(1 + \nu)$$

where $E/(2 + 2\nu)$ is the shear modulus, K is the stress intensity factor $(\varkappa = 3 - 4\nu$ for plane strain, $\varkappa = (3 - \nu)/(1 + \nu)$ for the plane state of stress, and ν is the Poisson's ratio) transforms condition (3.12) into the differential equation

$$\frac{K^2}{\mu} - l^{\cdot}\frac{2(1+\nu)}{E}\frac{\partial K^2}{\partial l} = 16(\varkappa + 1)\left(\xi + \frac{\partial\gamma}{\partial a} a^{\cdot}\right) \tag{3.13}$$

The roots of this equation determine the critical (fracture) values $K = K_c$. For $\gamma = \text{const}$ and viscous growth of a crack $K_c \sim \sqrt{\xi}$, i.e., K_c is a constant for $\xi = \text{const}$, but proportional to velocity $l^{\cdot}$ for $\xi \sim l^{\cdot 2}$.

For the extension of a linear viscoelastic strip with a crack $K = P\sqrt{\pi}\,(l/2)$ and equation (3.13) for $\gamma = \text{const}$ takes the form

$$2\xi - \frac{\pi P^2}{16(\varkappa+1)\mu} l + \frac{\pi P^2(1+\nu)}{8(\varkappa+1)E} l^{\cdot} = 0 \tag{3.14}$$

Comparing (3.14) and (2.3) shows that

$$\theta_1 = (\pi/16)(\varkappa+1)^{-1}, \quad \theta_2 = (\pi/8)(\varkappa+1)^{-1}(1+\nu)$$

4. If the dissipation is localized at the crack vertex /3/, then $\Lambda = \xi = 0$ and condition (2.5) is sufficient, but the dependence $\gamma_*(l^{\cdot})$ should be considered. If the localized dissipation is viscous in nature, then it follows from the second order of the homogeneity of the dissipative function /7/ that

$$D \sim \gamma_* l^{\cdot} = \eta l^{\cdot 2} \tag{4.1}$$

where in general $\eta = \eta(l^{\cdot}\tau/a)$, $\tau = \mu/E$. For viscoplastic Bingham media $\eta = \eta(Y/E)$, the resistance factor (in the terminology of /7/), turns out to be a material constant. Here Y is the yield point.

For a Bingham plane with a normal rupture crack of length l stretched by forces P, we have $G = \pi(P^2/E)(l/4)$. Condition (2.5) results in the following differential equation of quasi-static crack growth:

$$l^{\cdot}\left(l^{\cdot} - \frac{\pi P^2}{4E\eta} l + \frac{\gamma_0}{\eta}\right) = 0 \tag{4.2}$$

Solutions for a crack at rest ($l^{\cdot} = 0$) and growing

$$l - l_G = (l_0 - l_G)\exp\left(\frac{\pi P^2}{4E\eta} t\right), \quad l_G = \frac{4E\gamma_0}{\pi P^2} \tag{4.3}$$

hence follow.

In other words, the role of the initial threshold length l_0 of the crack is played by the Griffith value l_G, where the exponent grows, in contrast to (2.4), as the tensile force P grows.

Williams /8/ proposed a solution for crack growth in a stretched viscoelastic plane (from Kelvin–Voight material) according to which the idea of localized dissipation was used

$$2D = \eta_0 l^{\cdot 2}(P/E)^2 \tag{4.4}$$

independent of the crack length. However, the dissipation level (4.4) will be higher, the higher the crack velocity. Comparing (4.1) and (4.4) shows that $2\eta = \eta_0(P/E)^2$. Insertion of this latter value of η into the solution (4.3) yields the result /8/ according to which the exponent turns out to be independent of P.

Power-law dependence of γ_* on the crack velocity $l^{\cdot}$ were assumed in /9/. Crack development in such real materials as polymers is made complicated by accompanying temperature effects, which results in complex dependences of the energy expenditures on the crack velocity /10/.

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